Arithmetic of the BC-system

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• The three Witt constructions

• The BC-endomotive and $W_0(\overline{\mathbb{F}_p})$.

• The completion $W(\overline{\mathbb{F}_p})$ and $p$-adic representations of the BC-system.

• The KMS theory and division relations of polylogarithms.

• Iwasawa theory and extension of the KMS theory to a covering of $\mathbb{C}_p$. 

The initial motivation to seek for this relation came from the discovery that the algebraic relations fulfilled by the operators $\sigma_n$ and $\tilde{\rho}_n$ of $\mathcal{H}_\mathbb{Z}$ and the relations of the Frobenius endomorphisms $F_n : \mathbb{W}_0(A) \to \mathbb{W}_0(A)$, $n \in \mathbb{N}$ and Verschiebung additive functorial maps $V_n : \mathbb{W}_0(A) \to \mathbb{W}_0(A)$, $n \in \mathbb{N}$, in the Witt construction are identical.
2 fundamental operators act on $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$:

1) $\tilde{\rho}_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, $\tilde{\rho}_n(e(\gamma)) = \sum_{n\gamma' = \gamma} e(\gamma')$

2) $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, $\sigma_n(e(\gamma)) = e(n\gamma)$

Compatibility relations

- $\sigma_{nm} = \sigma_n \circ \sigma_m$, $\tilde{\rho}_{mn} = \tilde{\rho}_m \circ \tilde{\rho}_n$, $\forall m, n \in \mathbb{N}$
- $\tilde{\rho}_m(\sigma_m(x)y) = x\tilde{\rho}_m(y)$, $\forall x, y \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$
- $\sigma_c \circ \tilde{\rho}_b(x) = (b, c) \tilde{\rho}_{b'} \circ \sigma_{c'}(x)$, $b' = \frac{b}{(b, c)}$, $c' = \frac{c}{(b, c)}$
  \[ \sigma_n \circ \tilde{\rho}_n(x) = nx, \quad \forall n \in \mathbb{N} \]
  \[ \sigma_n \circ \tilde{\rho}_m = \tilde{\rho}_m \circ \sigma_n, \quad \text{if} \ (m, n) = 1 \]
The Witt functor $\mathbb{W}_0$

$\mathbb{W}_0(A)$ is functorially defined as the Grothendieck group of the category $\text{End}_A$ whose objects are pairs $(E, f)$ where $E$ is a finite projective module over $A$ and $f \in \text{End}_A(E)$ is an endomorphism of $E$.

$$(E_1, f_1) \oplus (E_2, f_2) = (E_1 \oplus E_2, f_1 \oplus f_2),$$

$$(E_1, f_1) \otimes (E_2, f_2) = (E_1 \otimes E_2, f_1 \otimes f_2)$$

turn the Grothendieck group $K_0(\text{End}_A)$ into a (commutative) ring. The pairs of the form $(E, f = 0)$ generate the ideal $K_0(A) \subset K_0(\text{End}_A)$.

$$\mathbb{W}_0(A) = K_0(\text{End}_A)/K_0(A).$$
By construction $\mathbb{W}_0$ is a functor from the category $\text{Ring}$ of commutative rings with unit to itself. The key additional structures are given by

- The Teichmüller lift which is a multiplicative map $\tau : A \to \mathbb{W}_0(A)$.

- The Frobenius endomorphisms $F_n$ for $n \in \mathbb{N}$.

- The Verschiebung (shift) additive functorial endomorphisms $V_n$, $n \in \mathbb{N}$.

- The ghost components $\text{gh}_n : \mathbb{W}_0(A) \to A$ for $n \in \mathbb{N}$. 
(1) The Teichmüller lift is simply given by the map $A \ni f \mapsto (A, f)$.

(2) For $n \in \mathbb{N}$, the following operations on $\text{End}_{A}$ induce endomorphisms in $\mathcal{W}_0(A)$ which are the Frobenius endomorphisms

$$F_n(E, f) = (E, f^n).$$

(3) The Verschiebung maps $V_n$ are described by the following operation on matrices:

$$V_n(f) = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 & f \\ 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$
(4) The ghost components are given by

\[ \text{gh}_n : \mathbb{W}_0(A) \rightarrow A, \quad \text{gh}_n(E, f) = \text{Trace}(f^n). \]
One has

(a) \( F_n \circ V_n(x) = nx. \)

(b) \( V_n(F_n(x)y) = xV_n(y). \)

(c) If \((m, n) = 1\), \( V_m \circ F_n = F_n \circ V_m. \)

(d) For \( n \in \mathbb{N}, \)
\( V_n(x)V_n(y) = nV_n(xy). \)

(e) For \( n \in \mathbb{N}, \)
\( F_n(\tau(f)) = \tau(f^n). \)

(f) For \( n, m \in \mathbb{N}, \)
\( gh_n(F_m(f)) = gh_{nm}(f). \)

(g) \( gh_n(V_m(f)) = \begin{cases} mgh_{n/m}(f) & \text{if } m|n \\ 0 & \text{otherwise} \end{cases}. \)
The BC-system and $\mathcal{W}_0(\overline{F}_p)$

**Theorem**

To each $\sigma : \overline{F}_p^\times \sim (\mathbb{Q}/\mathbb{Z})^\prime(p) \subset \mathbb{Q}/\mathbb{Z}$, corresponds a canonical isomorphism $\tilde{\sigma}$

$$
\begin{align*}
\mathcal{W}_0(\overline{F}_p) & \xrightarrow{\tilde{\sigma}} \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^\prime(p)] \subset \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\
& \xrightarrow{r=id\otimes \epsilon} \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^\prime(p)]
\end{align*}
$$

The Frobenius $F_n$ and Verschiebung maps $V_n$ of $\mathcal{W}_0(\overline{F}_p)$ are obtained by restriction of the endomorphisms $\sigma_n$ and maps $\tilde{\rho}_n$ of $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ by the formulas

$$
\tilde{\sigma} \circ F_n = \sigma_n \circ \tilde{\sigma}, \quad \tilde{\sigma} \circ V_n = r \circ \tilde{\rho}_n \circ \tilde{\sigma}
$$
This Theorem shows that the integral BC-system with its full structure is, if one drops the $p$-component, completely described as $\mathcal{W}_0(\overline{\mathbb{F}}_p)$. As a corollary one gets a representation $\pi_\sigma$ of the integral BC-system $\mathcal{H}_\mathbb{Z}$ on $\mathcal{W}_0(\overline{\mathbb{F}}_p)$,

$$
\pi_\sigma(x)\xi = \tilde{\sigma}^{-1}(r(x))\xi, \quad \pi_\sigma(\mu_n^*) = F_n, \quad \pi_\sigma(\tilde{\mu}_n) = V_n
$$

for all $\xi \in \mathcal{W}_0(\overline{\mathbb{F}}_p)$, $x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and $n \in \mathbb{N}$. 
Let $\Lambda(A) := 1 + tA[[t]]$. The characteristic polynomial defines a map

$$L : \mathbb{W}_0(A) \to \Lambda(A), \quad L(E, f) = \det(1_E - tf)^{-1},$$

and by a fundamental result of Almkvist, the map $L$ is always injective and its image is the subset of $\Lambda(A)$ whose elements are rational fractions

$$\text{Range}(L) = \{(1+a_1t+\ldots+a_nt^n)/(1+b_1t+\ldots+b_nt^n) \mid a_j, b_j \in A\}.$$
• The addition in \( \mathbb{W}_0(A) \) by
\[
L(f \oplus g) = L(f) L(g), \quad \forall f, g \in \mathbb{W}_0(A)
\]

• The Teichmüller lift which gives
\[
L(\tau(f)) = (1 - tf)^{-1} \in \Lambda(A)
\]

• The shift \( V_n \)
\[
L(V_n(f))(t) = L(f)(t^n)
\]

• The ghost components
\[
\sum_{1}^{\infty} gh_n(f) t^n = t \frac{d}{dt} \log(L(f(t))).
\]
This makes it clear how to extend the addition, the Teichmüller lift, the shifts and the ghost components to the completion \( \Lambda(A) \). The corresponding product \( * \) on \( \Lambda(A) \) is uniquely determined by functoriality and requiring that the ghost components define ring homomorphisms. It is given by explicit polynomials with integral coefficients of the form

\[
(1 + \sum a_nt^n) \ast (1 + \sum b_nt^n) = 1 + a_1b_1t + (a_1^2b_1^2 - a_2b_1^2 - a_1^2b_2 + 2a_2b_2)t^2 + (a_1^3b_1^3 - 2a_1a_2b_1^3 + a_3b_1^3 - 2a_1^3b_1b_2 + 5a_1a_2b_1b_2 - 3a_3b_1b_2 + a_1^3b_3 - 3a_1a_2b_3 + 3a_3b_3)t^3 + \ldots
\]
Every element \( f(t) \in \Lambda(A) \) can be written uniquely as an infinite product

\[
f(t) = \prod (1 - z_n t^n), \quad z_n \in A, \quad \forall n
\]

This shows that the following map is a bijection from the set \( \mathcal{W}(A) \) of sequences \((x_n)\) of elements of \( A \) to \( \Lambda(A) \)

\[
\varphi_A : \mathcal{W}(A) \to \Lambda(A) := 1 + tA[[t]],
\]

\[
x = (x_n)_{n \in \mathbb{N}} \mapsto f_x(t) = \prod_{n \in \mathbb{N}} (1 - x_n t^n)^{-1}.
\]
At the level of the Witt vector components $x_j$ the Frobenius $F_n$ is given by polynomials with integral coefficients and for instance the first 5 components of $F_3(x)$ are

\[
F_3(x)_1 = x_1^3 + 3x_3 \\
F_3(x)_2 = x_2^3 - 3x_1^3x_3 - 3x_3^2 + 3x_6 \\
F_3(x)_3 = -3x_1^6x_3 - 9x_1^3x_3^2 - 8x_3^3 + 3x_9 \\
F_3(x)_4 = -3x_1^9x_3 + 3x_1^3x_2^3x_3 - 18x_1^6x_3^2 + 3x_2^3x_3^2 - 36x_1^3x_3^3 - 24x_3^4 + x_4^3 - 3x_2^3x_6 + 9x_1^3x_3x_6 + 9x_3^2x_6 - 3x_6^2 + 3x_12 \\
F_3(x)_5 = -3x_1^{12}x_3 - 18x_1^9x_3^2 - 54x_1^6x_3^3 - 81x_1^3x_3^4 - 48x_3^5 + x_5^3 + 3x_1
\]
Artin-Hasse exponential

\[ E_p(t) = \text{hep}(t) = \exp(t + \frac{t^p}{p} + \frac{t^{p^2}}{p^2} + \cdots) \in \Lambda(\mathbb{Z}_p). \]

(1) \( E_p(t) \) is an idempotent of \( \Lambda(\mathbb{Z}_p) \).

(2) For \( n \in I(p) \), the series \( E_p(n)(t) := \frac{1}{n} V_n(E_p)(t) \in \Lambda(\mathbb{Z}_p) \) determine an idempotent. As \( n \) varies in \( I(p) \), the \( E_p(n) \) form a partition of unity by idempotents.

(3) For \( n \notin p^\mathbb{N} \), \( F_n(E_p)(t) = 1(= 0_\Lambda) \) and \( F_{p^k}(E_p)(t) = E_p(t), \forall k \in \mathbb{N}. \)
(a) The map
\[ \psi_A: \mathbb{W}_p^\infty(A) \to \Lambda(A)_E, \quad \psi_A(x)(t) := h_x(t) = \prod_{n} E_p(x p^n t^p) \]
is an isomorphism onto the reduced ring \( \Lambda(A)_E = \{ x \in \Lambda(A) \mid x \star E_p = x \} \).

(b) The composite
\[ \theta_A(x) = (\theta_A(x))_n = \psi_A^{-1} \circ F_n(x \star E_p(n)), \quad n \in I(p), \ x \in \Lambda(A) \]
is a canonical isomorphism \( \theta_A: \Lambda(A) \to \mathbb{W}_p^\infty(A)^{I(p)} = \mathbb{W}(A) \).

(c) The composite isomorphism \( \Theta_A := \theta_A \circ \varphi_A: \mathbb{W}(A) \to \mathbb{W}_p^\infty(A)^{I(p)} \) is given explicitly on the components by
\[ (\Theta_A(x)_n)_{p^k} = F_n(x)_{p^k}, \quad \forall x \in \mathbb{W}(A), \quad \forall n \in I(p). \]
The completion $W(F_p)$ of $W_0(F_p)$

One has the isomorphism

$$W(F_p) = (W_{p^\infty}(\overline{F}_p))^I(p)$$

where $W_{p^\infty}$ is the Witt functor using the set of powers of the prime $p$.

Let $(\widehat{\mathbb{Q}^\text{un}}_p) \subset \mathbb{C}_p$ be the completion of the maximal unramified extension $\mathbb{Q}^\text{un}_p$ of $p$-adic numbers. Then $W_{p^\infty}(\overline{F}_p)$ is the completion $\mathcal{O} = \mathcal{O}_{\mathbb{Q}^\text{un}_p} \subset (\widehat{\mathbb{Q}^\text{un}}_p)$ of the subring generated by roots of unity.
The $p$-adic representations of $\mathcal{H}_{\mathbb{Z}}$

**Theorem**

Let $\sigma \in X_p$ and $\rho : \mathbb{Q}^{\text{cycl},p} \to \mathbb{C}_p$ associated to $\sigma$. The representation $\pi_\sigma$ of $\mathcal{H}_{\mathbb{Z}}$ extends to $\mathbb{W}(\overline{\mathbb{F}}_p)$. For $n \in I(p)$, the $\pi_\sigma(\mu_n)$ and for $x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, the $\pi_\sigma(x)$ are $\mathcal{O}$-linear and

$$
\pi_\sigma(\mu_n)\epsilon_m = \epsilon_{nm}, \quad \pi_\sigma(e(a/b))\epsilon_m = \rho(\zeta_{a/b}^m)\epsilon_m
$$

$$
\forall n \in \mathbb{N}, m, b \in I(p).
$$

Moreover

$$
\pi_\sigma(\mu_p) = \text{Fr}^{-1}
$$

is the inverse of the Frobenius.
Ideal $\mathcal{J}_p$ of $\mathcal{H}_\mathbb{Z}$

Proposition

Let $\mathcal{J}_p \subset \mathcal{H}_\mathbb{Z}$ be the two-sided ideal generated by

$$1 - e(p^{-k}) , \quad \forall k \in \mathbb{N}$$

Then

$$\mathcal{J}_p = \text{Ker}(\pi_\sigma)$$

and $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \cap \mathcal{J}_p$ is the ideal $\mathcal{J}_p^0$ of $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ generated by the $1 - e(p^{-k})$.

$$0 \rightarrow \mathcal{J}_p^0 \rightarrow \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{(p)}] \rightarrow 0$$

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The algebra $\mathcal{H}_Z^{(p)}$

Let $\mathcal{H}_Z^{(p)}$ quotient by $\mathcal{J}_p$ of subalgebra of $\mathcal{H}_Z$ generated by $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{(p)}]$, $\tilde{\mu}_n$, $\mu_n^*$, $n \in I(p)$.

It is generated by $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{(p)}]$, $\tilde{\mu}_n$, $\mu_n^*$ for $n \in I(p)$, the relations are

\[\tilde{\mu}_{nm} = \tilde{\mu}_n \tilde{\mu}_m, \quad \mu_{nm}^* = \mu_n^* \mu_m^*, \quad \forall n, m \in I(p)\]

\[\mu_n^* \tilde{\mu}_n = n, \quad \forall n \in I(p)\]

\[\tilde{\mu}_n \mu_m^* = \mu_m^* \tilde{\mu}_n, \quad \forall n, m \in I(p), \ (n, m) = 1.\]

\[\tilde{\mu}_n x \mu_n^* = \tilde{\rho}_n(x), \quad \mu_n^* x = \sigma_n(x) \mu_n^*, \quad x \tilde{\mu}_n = \tilde{\mu}_n \sigma_n(x),\]

where $\tilde{\rho}_n$, $n \in I(p)$

\[\tilde{\rho}_n(e(\gamma)) = \sum_{n\gamma' = \gamma} e(\gamma'), \quad \forall \gamma \in \mu^{(p)}.\]
**Automorphism** $\text{Fr} \in \text{Aut}(\mathcal{H}_\mathbb{Z}^{(p)})$

Unique $\text{Fr} \in \text{Aut}(\mathcal{H}_\mathbb{Z}^{(p)})$ such that

$$\text{Fr}(e(\gamma)) = e(\gamma)^p, \quad \forall \gamma \in \mu^{(p)},$$

$$\text{Fr}(\tilde{\mu}_n) = \tilde{\mu}_n, \quad \text{Fr}(\mu^*_n) = \mu^*_n, \quad \forall n \in I(p)$$

One has an isomorphism

$$\mathcal{H}_\mathbb{Z}/\mathcal{J}_p = \mathcal{H}_\mathbb{Z}^{(p)} \rtimes \text{Fr},_p \mathbb{Z}$$

$\mathcal{A} \rtimes_{\theta, p} \mathbb{Z} \subset \{ \sum \mathbb{Z} a_n V^n \mid a_n \in \mathcal{A} \}$ by the condition:

$$a_{-n} \in p^n \mathcal{A}, \quad \forall n \in \mathbb{N}$$

Passing from $\mathcal{H}_\mathbb{Z}^{(p)}$ to $\mathcal{H}_\mathbb{Z}$ is given by covariance for the action of the Frobenius $\text{Fr}$ of $\mathcal{O} = \mathcal{O}_{\mathbb{Q}_p}^{\text{un}}$. 

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(1) The restriction $\pi_\sigma|_{\mathcal{H}_\mathbb{Z}^{(p)}}$ of the representation $\pi_\sigma$ to $\mathcal{H}_\mathbb{Z}^{(p)}$ is $\mathcal{O}$-linear and indecomposable over $\mathcal{O}$.

(2) The representations $\pi_\sigma|_{\mathcal{H}_\mathbb{Z}^{(p)}}$ are pairwise inequivalent.

(3) The representation $\pi_\sigma$ is linear and indecomposable over $\mathbb{Z}_p$.

(4) Two representations $\pi_\sigma$ and $\pi_{\sigma'}$ are equivalent over $\mathbb{Z}_p$ if and only if there exists $\alpha \in \text{Aut}(\overline{\mathbb{F}}_p)$ such that $\sigma' = \sigma \circ \alpha$. 
The KMS theory and division relations of polylogs

These representations are the $p$-adic analogues of the complex, extremal KMS$_\infty$ states of the BC-system. We have pursued this analogy much further by implementing the Iwasawa theory of $p$-adic $L$-functions to construct, in the $p$-adic case, the partition function and the KMS$_\beta$ states. In particular, we have shown that the division relations for the $p$-adic polylogarithms at roots of unity correspond to the KMS condition.
The time evolution in the $p$-adic case

$q = 4$, if $p = 2$, $q = p$, if $p \neq 2$.

$$D_p = \{ \beta \in \mathbb{C}_p \mid |\beta|_p < q p^{-1/(p-1)} > 1 \} ,$$

Let $r \in \mathbb{Z}_{(p)}^\times$. There exists a unique analytic function

$$D_p \to \mathbb{C}_p, \quad \beta \mapsto r(\beta)$$

such that

$$r(\beta) = r^\beta, \quad \forall \beta = 1 - k \varphi(q).$$

$$r(\beta) := r \exp((\beta - 1) \log_p(r)), \quad \forall \beta \in D_p.$$

$$\mathbb{Z}_{(p)}^\times \ni r \mapsto r(\beta) \in \mathbb{C}_p^\times$$

is a group homomorphism.
The automorphisms $\sigma^{(\beta)} \in \text{Aut}(\mathcal{H}_{\mathbb{C}^p})$

(1) For $\beta \in D_p$ there exists a unique automorphism $\sigma^{(\beta)} \in \text{Aut}(\mathcal{H}_{\mathbb{C}^p})$ such that

$$\sigma^{(\beta)}(\tilde{\mu}_a e(\gamma)\mu_b^*) = \left(\frac{b}{a}\right)^{(\beta)} \tilde{\mu}_a e(\gamma)\mu_b^*, \quad \forall a, b \in I(p), \gamma \in (\mathbb{Q}/\mathbb{Z})^{(p)}.$$ 

(2) One has

$$\sigma^{(\beta_1)} \circ \sigma^{(\beta_2)} = \sigma^{(\beta_1 + \beta_2)} \circ \sigma^{(0)}, \quad \forall \beta_j \in D_p$$

and $\sigma^{(0)}$ is an automorphism of order $\varphi(q)$. 

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One looks for a linear form

$$\phi : \mathcal{H}_{\mathbb{C}_p}^{(p)} \to \mathbb{C}_p, \quad \phi(1) = 1,$$

such that

$$\phi(x \sigma_\beta(y)) = \phi(y x), \quad \forall x, y \in \mathcal{H}_{\mathbb{C}_p}^{(p)}.$$
Formal expression in the $p$-adic case

One looks for

$$Z\left(\frac{a}{b}, \beta\right) = \sum_{m \in I(p)} \rho(\zeta_{a/b}^m) m^{-\beta}, \quad \beta \in D_p$$

$\rho(\zeta_{a/b}^m)$ only depends on $m$ modulo $b$. Let $f = bp$, one decomposes the sum in terms of the residue $c$, $c < f$, of $m$ modulo $f$.

$$Z\left(\frac{a}{b}, \beta\right) = \sum_{1 \leq c < bp} \rho(\zeta_{a/b}^c) \sum_{n \in \mathbb{N}} (c + fn)^{-\beta}$$

$$\quad \sum_{1 \leq c < bp} \rho(\zeta_{a/b}^c) \frac{f^{-\beta}}{\mathbb{N}} \sum_{n \in \mathbb{N}} \left(\frac{c}{f} + n\right)^{-\beta}$$
For $z = \frac{c}{f}$, one has $c \notin p\mathbb{N}$ and $|z^{-1}|_p < 1$.

$$\sum_{n \in \mathbb{N}} (z + n)^{-\beta} = \frac{z^{1-\beta}}{-1 + \beta} \sum_{j=0}^{\infty} \binom{1-\beta}{j} B_j z^{-j}$$

$$f^{-\beta} \sum_{n \in \mathbb{N}} (z + n)^{-\beta} = \frac{1}{f} \frac{c^{1-\beta}}{\beta - 1} \sum_{j=0}^{\infty} \binom{1-\beta}{j} B_j z^{-j}$$
Euler-Maclaurin formula

\[ \sum_{k=a}^{b} f(k) = \int_{a}^{b} f(t) dt + \frac{f(a) + f(b)}{2} \]

\[ + \sum_{j=2}^{m} \frac{B_j}{j!} (f^{(j-1)}(b) - f^{(j-1)}(a)) - R_m \]

\[ R_m = \frac{(-1)^{m}}{m!} \int_{a}^{b} f^{(m)}(x) B_m(x - [x]) dx \]

\[ \sum_{k=0}^{\infty} f(k) \sim \int_{0}^{\infty} f(t) dt + \frac{1}{2} f(0) - \sum_{j=2}^{\infty} \frac{B_j}{j!} f^{(j-1)}(0) \]
One takes \( f(x) = (z + x)^{-s} \).

\[
f^{(k)}(0) = k! \binom{-s}{k} z^{-s-k}, \quad \int_0^\infty f(x)dx = \frac{z^{1-s}}{-1+s}\]

\[
-\frac{B_j}{j!} f^{(j-1)}(0) = \frac{z^{1-s}}{-1+s} \binom{1-s}{j} B_j z^{-j}
\]

For \( j = 0 \) one gets \( \frac{z^{1-s}}{-1+s} \) contribution of \( \int_0^\infty f(t)dt \), for \( j = 1 \)

\[
\frac{z^{1-s}}{-1+s} \binom{1-s}{1} \left(-\frac{1}{2}\right) z^{-1} = \frac{1}{2} z^{-s}
\]
contribution of $\frac{1}{2} f(0)$. One gets

\[\sum_{k=0}^{\infty} (z + k)^{-s} \sim \frac{z^{1-s}}{-1 + s} \sum_{j=0}^{\infty} \binom{1-s}{j} \sum_{j} B_j z^{-j}\]
Construction of $Z(a/b, \beta)$

One defines

$$Z(a/b, \beta) := \frac{1}{bq} \sum_{1 \leq c < bq} \rho(\zeta_{a/b}^{c}) \frac{\langle c \rangle^{1-\beta}}{\beta - 1} \sum_{j=0}^{\infty} \left(1 - \beta \right)^{j} \left(\frac{bp}{c}\right)^{j} B_{j}$$

**Lemma** The function $Z(a/b, \beta)$ is meromorphic with a simple pole at $\beta = 1$ in $D_{p}$.
Theorem

For any $\beta \in D_p$, $\beta \neq 1$, and $\rho \in \text{Hom}(\mathbb{Q}^{\text{cycl}}, p, \mathbb{C}_p)$ the linear form $\varphi_{\beta, \rho}$ fulfills the KMS$_\beta$ condition:

$$\varphi_{\beta, \rho}(x \sigma(\beta)(y)) = \varphi_{\beta, \rho}(yx), \quad \forall x, y \in \mathbb{H}_c^{(p)}.$$

The partition function is the $p$-adic $L$-function for the character $\chi = 1$,

$$Z(\beta) = L_p(\beta, 1).$$

One has $Z(\beta) \neq 0$ for $\beta \in D_p$ and a pole at $\beta = 1$ with residu $\frac{p-1}{p}$. 

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Partition function

\[ \int \langle a \rangle^\beta d\eta = \left( 1 - (1 + q)^{1-\beta} \right) L_p(\beta, 1) \]

There exists \( \eta(T) \in O_p[[T]] \) such that

\[ ((1 + q)^{1-\beta} - 1)Z(\beta) = \eta((1 + q)^\beta - 1) \]

\[ \frac{1}{2} \eta(T) \in O_p[[T]]^\times \]
Change of parameter $\beta \mapsto \lambda$

$$((1 + q)^{1-\beta} - 1)Z(\beta) = \eta((1 + q)^\beta - 1)$$

shows that $\lambda = (1 + q)^\beta$ is the right parameter.

$$M = D(1, 1^-) = \{\lambda \in \mathbb{C}_p \mid |\lambda - 1|_p < 1\}$$

open disk and multiplicative group. One has a homomorphism

$$\lambda \in M = D(1, 1^-) \mapsto \beta = \frac{\log_p \lambda}{\log_p (1 + q)} \in \mathbb{C}_p$$

surjective with kernel $\mu_{p\infty}$. 
Extension to the covering

**Theorem** There exists an analytic family of functionals $\psi_{\lambda,\rho}$, $\lambda \in M$, on $\mathcal{H}_{Z}^{(p)}$ such that

1. $\psi_{\lambda,\rho}(1) = 1$.
2. $\psi_{\lambda,\rho}$ fulfills the KMS condition
   \[
   \psi_{\lambda,\rho}(x\sigma[\lambda](y)) = \psi_{\lambda,\rho}(yx), \quad \forall x, y \in \mathcal{H}_{\mathbb{C}p}^{(p)}.
   \]
3. For $\beta \in D_{p}$ and $\lambda = (1 + q)^{\beta}$ one has
   \[
   \psi_{\lambda,\rho} = Z(\beta)^{-1}\varphi_{\beta,\rho}.
   \]

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For $\beta \in D_p$ and $r \in \mathbb{Z}_p^\times$ one has

\[ r^{(\beta)} = \omega(r) \lambda_i^p(r), \quad \langle r \rangle^\beta = \lambda^i_p(r) \]

\[ i_p(r) = \frac{\log_p(r)}{\log_p(1 + q)} \in \mathbb{Z}_p, \quad \lambda = (1 + q)^\beta \]
One compares

$$Z_\rho\left(\frac{a}{b}, \beta\right) := \frac{1}{f} \sum_{1 \leq c < f \atop c \notin pN} \rho(\zeta_{a/b}^c) \frac{\langle c \rangle^{1-\beta}}{\beta - 1} \sum_{j=0}^{\infty} \left(1 - \beta\right) \left(\frac{f}{c}\right)^j B_j$$

with \(p\)-adic \(L\)-functions

$$L_p(\beta, \chi) := \frac{1}{f} \sum_{1 \leq c < f \atop c \notin pN} \chi(c) \frac{\langle c \rangle^{1-\beta}}{\beta - 1} \sum_{j=0}^{\infty} \left(1 - \beta\right) \left(\frac{f}{c}\right)^j B_j$$

where \(\chi\) is a primitive Dirichlet character.
Lemma

Let $a/b \in (\mathbb{Q}/\mathbb{Z})^{(p)}$, there exists $c(d, \chi) \in \mathbb{C}_p$ such that

$$Z_\rho\left(\frac{a}{b}, \beta\right) =$$

$$\sum_{d|b} c(d, \chi)L_p(\beta, \chi)d^{-1}\langle d \rangle^{1-\beta} \prod(1 - \chi(\ell)\ell^{-1}\langle \ell \rangle^{1-\beta})$$

where $d$ divides $b$, and $\chi$ is a primitive character of conductor $f$ dividing $m = b/d$ and the $\ell$ are the prime divisors of $m/f$ prime to $f$. 
Let $\chi$ be a primitive Dirichlet character with conductor prime to $p$. If $\chi$ is odd ($\chi(-1) = -1$) one has $L_p(\beta, \chi) = 0$.

If $\chi$ is even there exists $h_\chi \in \mathcal{O}_p[[T]]$ such that

$$L_p(\beta, \chi) = h_\chi((1 + q)^\beta - 1)$$
Case $\beta = 1$

when $\beta \to 1$ one has

$$Z(\beta)^{-1}Z\rho\left(\frac{a}{b}, \beta\right) \rightarrow \begin{cases} 1, & \text{if } \frac{a}{b} \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

One obtains the regular representation

$$\sigma_{(1)}(\tilde{\mu}_a e(\gamma) \mu_b^*) = \frac{b}{a} \tilde{\mu}_a e(\gamma) \mu_b^*$$
Lemma:

Let $\lambda \in \hat{\mathbb{Z}}^*$, $\lambda \neq \pm 1$. The graph of multiplication by $\lambda$ in $\mathbb{Q}/\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$.

Let $\theta \in \text{Aut}(\mu^{(p)})$, $\theta \notin \{\pm p\mathbb{Z}\}$. The graph of $\theta$ is dense in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$.
Let $\lambda \in \hat{\mathbb{Z}}^*$

$$G = \{(\alpha, \lambda \alpha) \mid \alpha \in \mathbb{Q}/\mathbb{Z}\}$$

subgroup of $T = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. If $\bar{G} \neq T$ there exists $(n, m) \in \mathbb{Z}^2$ non zero such that $G$ is in the kernel of the character $(n, m)$,

$$n\alpha + m\lambda \alpha \in \hat{\mathbb{Z}}, \quad \forall \alpha \in \mathbb{A}_\mathbb{Q}, f$$

$$(n + m\lambda_p)\alpha \in \mathbb{Z}_p, \quad \forall \alpha \in \mathbb{Q}_p$$

et $n + m\lambda_p = 0$ for all $p$.

In the second case

$$\text{Aut}(\mu^{(p)}) = \prod_{\ell \neq p} \mathbb{Z}_\ell^*$$

$n + m\lambda_\ell = 0$ for $\ell \neq p$ and $-n/m \in \{\pm p\mathbb{Z}\}$, $\theta \in \{\pm p\mathbb{Z}\}$

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Thus if \( f : \{ z \in \mathbb{C} \mid |z| = 1 \} \to \mathbb{C} \) is continuous non-constant, and \( \rho_j : \mathbb{Q}^{\text{cycl}} \to \mathbb{C} \), an equality

\[
f(\rho_1(\zeta_{a/b})) = f(\rho_2(\zeta_{a/b})), \quad \forall a/b \in \mathbb{Q}/\mathbb{Z}
\]

implies \( \rho_2 = \rho_1 \) or \( \rho_2 = \bar{\rho}_1 \). This case

\[
f(\bar{z}) = f(z), \quad \forall z, \ |z| = 1.
\]
cannot happen for \( f(z) = \sum_{n=1}^{\infty} n^{-\beta}z^n, \ \Re(\beta) > 1 \).
Let $\theta \in \text{Aut}(\mu^{(p)})$. If $\theta \in \{\pm 1\}$ one has

$$Z_\rho\left(\frac{a}{b}, \beta\right) = Z_{\theta \circ \rho}\left(\frac{a}{b}, \beta\right), \quad \forall a/b \in \mu^{(p)}, \beta \in D_p.$$

If $\theta \notin \{\pm 1\}$ and $\beta = 1 - m = 1 - k\varphi(q)$, the functionals $Z_\rho(\cdot, \beta)$ and $Z_{\theta \circ \rho}(\cdot, \beta)$ are distincts.

$$\sum_{1 \leq c \leq b} \rho'(\zeta_{a/b}^c) B_m\left(\frac{c}{b}\right) = \sum_{0 \leq c \leq b - 1} \rho(\zeta_{a/b}^c) B_m\left(\frac{b - c}{b}\right).$$

$$B_m(1 - x) = B_m(x), \quad \forall m \in 2\mathbb{N}.$$
Let $Fr \in \text{Aut}((\overline{\mathbb{Q}_p^{un}}))$ be the Frobenius, one has

$$(1 - p^{-\beta}Fr)^{-1} Z_\rho \left( \frac{a}{b}, \beta \right)$$

$$= -\frac{b^{m-1}}{m} \sum_{1 \leq c \leq b} \rho(\zeta_{a/b}^c) B_m \left( \frac{c}{b} \right) \in \overline{\mathbb{Q}_p^{un}}$$

and the equality $Z_\rho = Z_{\rho'}$ implies

$$\sum_{1 \leq c \leq b} \rho(\zeta_{a/b}^c) B_m \left( \frac{c}{b} \right) = \sum_{1 \leq c \leq b} \rho'(\zeta_{a/b}^c) B_m \left( \frac{c}{b} \right)$$

$$B_m(\theta^{-1}(\frac{c}{b})) = B_m \left( \frac{c}{b} \right) \quad \forall c/b \in \mu(p).$$

$$B_m(x) = B_m(p^a x - [p^a x]) \quad \forall x \in [0, 1]$$

would imply $B_m(x) - B_m(p^a x) = 0$. 
