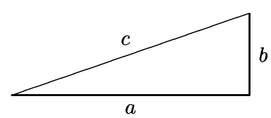
pages	$2-5$	$6-10$	$11-14$	$15-18$	$19-23$	$24-27$	*total*
scores							

Final Exam May 6, Linear Algebra, Spring, 2004, W. Stephen Wilson

Name: ______________________________

TA Name and section: ______________________________

NO CALCULATORS, SHOW ALL WORK, NO SCRAP PAPER.



The Pythagorean Theorem for a right triangle (see picture above) says that $c^2 = a^2 + b^2$. This has nothing to do with the course (in particular, it has nothing to do with your grade) but because of my interest in K-12 education I'd like to take just a minute of your time to ask you a question.

Circle all that apply:

(1) If you never heard of the theorem in K-12.
(2) If you heard of it but didn't need to use it.
(3) If you heard of it and had to use it.
(4) If you saw the proof of it.
(5) If you had to learn how to prove it.
(6) You think you can still prove it.
(7) Can't remember.
(8) Other and/or comment: ______________________________

1 2 3 4 5 6 7

The matrix $B = \begin{pmatrix} 5/2 & 0 & 3/2 \\ 0 & 2 & 0 \\ 3/2 & 0 & 5/2 \end{pmatrix}$ will be used for several problems. The numbers tend to work out nicely. Be careful with your arithmetic.

(1) (3 points) What is the determinant of B?

(2) (3 points) What is the rank of B?

(3) (3 points) Find the inverse of B, B^{-1}.

(4) (3 points) Check the inverse of B on one side.

(5) (3 points) What is the characteristic polynomial of B?

(6) (3 points) What are the Eigenvalues for B?

(7) (3 points) Find an orthonormal Eigenbasis for B. Order the basis by taking the Eigenvector for the biggest Eigenvalue first and so on.

(8) (3 points) What is the change of basis matrix for our new basis in the previous problem?

(9) (3 points) Find the inverse of the change of basis matrix in the previous problem.

(10) (3 points) The last section of the book that we did produced a beautiful description of an arbitrary linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$. In the case where $n = 3 = m$, describe what happens. (You need vectors $v_i \in \mathbb{R}^3$ (for $n = 3$) and $u_i \in \mathbb{R}^3$ (for $m = 3$).)

We will now use the matrix $A = \begin{pmatrix} 1/2 & 1 & -1/2 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1/2 & -1 & -1/2 \end{pmatrix}$.

(11) (3 points) What are the singular values of A?

(12) (3 points) Find the vectors v_i of the theorem of problem 10 for A of problem 11.

(13) (3 points) Find the vectors u_i of the theorem of problem 10 for A of problem 11.

(14) (3 points) Now, more explicitly, state the result for our matrix A of problem 11.

(15) (3 points) State the singular-value decomposition (SVD) for an $m \times n$ matrix C.

(16) (3 points) Give the singular-value decomposition for the matrix A of problem 11 we have been studying.

We work in $P_1 \subset P_2 \subset C[-1,0]$ with the usual inner product.

(17) (3 points) What is the unsual inner product?

(18) (3 points) Find an orthonormal basis for $P_1 \subset C[-1,0]$.

(19) (3 points) What is the orthogonal projection of $x^2 \in P_2 \subset C[-1, 0]$ onto $P_1 \subset P_2$, i.e. $\text{proj}_{P_1}(x^2)$? (Show work.)

(20) (3 points) Using the basis $\{x^2, x, 1\}$ for P_2, find the 3×3 matrix for $\text{proj}_{P_1} : P_2 \to P_1 \subset P_2$.

(21) (3 points) What is the dimension of the kernel of this linear transformation (in problem (19-20))? (Explain.)

(22) (3 points) Find a polynomial basis for the kernel in (21).

(23) (3 points) The polynomial of problem (19) minimizes a certain *least squares* integral. What is that integral?

(24) (3 points) Find the area of the parallelogram determined by the two vectors $(1, 1, 1)$ and $(1, 1, 0)$. (Lots of ways to do this.)

(25) (3 points) If the determinant of $(v_1, v_2, v_3) = 1$, what is the determinant of (v_3, v_1, v_2)?

(26) (3 points) If the determinant of $(v_1, v_2, v_3) = 1$, what is the determinant of $(v_1 + v_3, v_1 + v_3, v_2 + v_1)$?

(27) (3 points) If the determinant of $(v_1, v_2, v_3) = 1$, what is the determinant of $(3v_1, v_1 + 3v_3, v_2 + 3v_1)$?

(28) (3 points) Find an orthonormal basis for $\mathbb{R}^4$ that includes the vectors $(1, 0, 1, 1)/\sqrt{3}$ and $(1, 1, -1, 0)/\sqrt{3}$.

(29) (3 points) Let $\{v_1, v_2, v_3, v_4\}$ be a basis for $\mathbb{R}^4$ and let $L : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear tranformation that takes v_1 to $5v_1 + 3v_3$, v_2 to $v_2 + 2v_4$, v_3 to $v_1 - v_2$, and v_4 to $v_2 - v_1$. Find the matrix representing L with respect to this basis.

(30) (3 points) Find a basis for the kernel of the linear transformation in problem (29). (Write it as a linear combination of the v_i.)

We now use the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ in the next few problems.

(31) (3 points) Find a basis for the kernel of A.

(32) (3 points) Find an orthonormal basis for the kernel of A.

(33) (3 points) Find a basis for the orthogonal complement of the kernel of A.

(34) (3 points) Find an orthonormal basis for the orthogonal complement of the kernel of A.

(35) (3 points) Find a basis for the image of A.

(36) (3 points) Find an orthonormal basis for the image of A.

(37) (3 points) Find the points (2) in $\mathbb{R}^3$ on $(5/2)x_1^2 + 2x_2^2 + (5/2)x_3^2 + 3x_1x_3 = 1$ closest to the origin.

(38) (3 points) How far are the points of problem 37 from the origin?

(39) (3 points) Find a length one vector in the subspace of $C[0, 1]$ (with the usual inner product) spanned by x^2.