Final Exam, Linear Algebra, Spring, 2003, W. Stephen Wilson

Name: ______________________________

TA Name and section: ______________________________

NO CALCULATORS.

(1) (3 points) Give a definition of *linear independence*.

(2) (3 points) If $A : \mathbb{R}^n \to \mathbb{R}^n$, there is a relationship between the dimension of the kernel of A and the dimension of the image of A. What is it?

(3) (3 points) Define *perpendicular* in $\mathbb{R}^n$.

(4) (3 points) Define the *length* of a vector in $\mathbb{R}^n$.

(5) (3 points) Find a basis for the orthogonal complement (in $\mathbb{R}^4$) of the subspace spanned by

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

We will now be working with the matrix $A = \begin{pmatrix} 3 & 0 & -3 \\ -1 & 4 & 9 \\ 0 & -2 & -4 \end{pmatrix}$ for some time now.

(6) (3 points) What is the trace of A above?

(7) (3 points) What is the determinant of A above?

(8) (3 points) Solve the equation $Ax = 0$ for the A above.

(9) (3 points) For the A above, solve the equation $Ax = \begin{pmatrix} 0 \\ 8 \\ -4 \end{pmatrix}$.

(10) (3 points) Find a basis for the kernel of the A above.

(11) (3 points) Find a basis for the row space of the A above.

(12) (3 points) What is the rank of the A above?

(13) (3 points) Find a basis for the image of the A above.

(14) (3 points) Find the characteristic polynomial for the A above.

(15) (3 points) Find the Eigenvalues for the A above. (They are all integers.)

(16) (3 points) For the A above, find an Eigenvector for each of the Eigenvalues. To make it easier to grade, choose Eigenvectors with integer coordinates where the integers are as small as possible.

(17) (3 points) Use your Eigenvectors to make a basis of $\mathbb{R}^3$. Chose the first basis vector to be the Eigenvector associated with the largest Eigenvalue and the third basis vector to be the Eigenvector associated with the smallest Eigenvalue. Call this basis $\mathcal{B}$. We have a linear transformation given to us by A. What is the matrix B when we use coordinates from this new basis, i.e. $B : [x]_{\mathcal{B}} \to [Ax]_{\mathcal{B}}$?

(18) (3 points) We know there is a matrix S such that $S^{-1}AS = B$. Find S.

(19) (3 points) Find S^{-1}.

(20) (3 points) Let P_3 be all polynomials with degree less than or equal to 3. Let $C[-1,1]$ be the continuous functions from the interval $[-1,1]$ to the reals. $P_3 \subset C[-1,1]$. They have an inner product given by $< f, g >= \int_{-1}^{1} f(x)g(x)dx$. Show the set $V \subset P_3 \subset C[-1,1]$ defined as all $f \in P_3$ with $f(-1) = 0$ and $\int_{-1}^{1} f(x)dx = 0$ is a linear subspace of P_3.

(21) (3 points) Find a basis for the linear space in (20).

We work in $P_1 \subset P_2 \subset C[0,1]$ with the usual inner product ($< f, g >= \int_0^1 f(x)g(x)dx$). Recall that P_n is the set of polynomials of degree less than or equal to n. It is a fact that an orthonormal basis for P_1 is given by $\{1, \sqrt{3}(2x-1)\}$. You can now assume that.

(22) (3 points) What is the orthogonal projection of $x^2 \in P_2$ onto $P_1 \subset P_2$, i.e. $\text{proj}_{P_1}(x^2)$? (Show work.)

(23) (3 points) Using the basis $\{x^2, x, 1\}$ for P_2, find the 3×3 matrix for $\text{proj}_{P_1} : P_2 \to P_1 \subset P_2$.

(24) (3 points) What is the dimension of the kernel of this linear transformation (in problem (23))? (Explain.)

(25) (3 points) Find a polynomial basis for the kernel in (24).

(26) (3 points) Find an orthogonal (not necessarily orthonormal) basis for P_2 that extends the known orthogonal basis for P_1 from before problem (22).

(27) (3 points) The polynomial of problem (22) minimizes a certain *least squares* integral. What is that integral?

(28) (3 points) For the equation $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find all of the least squares solutions.

(29) (3 points) If $\det(A) = 5$ and A is an $n \times n$ matrix, then what is $\det(5A)$? (Explain a bit.)

(30) (3 point) If A is an orthogonal rotation $n \times n$ matrix, then what is $\det(5A)$? (Explain a bit.)

(31) (3 points) Write the quadratic form x_1x_2 in matrix form with a symmetric matrix.

(32) (3 points) Say whether $x_1x_2 = 1$ is an ellipse or a hyperbola. Justify using the approach in the course.

(33) (3 points) Find the principal axes for $x_1 x_2 = 1$ and locate the intercepts.

(34) (3 points) Give the form of the curve $x_1 x_2 = 1$ in the coordinate system defined by the principal axes.

(35) (3 points) State Cramer's rule.

(36) (3 points) What is the area of the parallelogram defined by the vectors $(1, 5)$ and $(3, 9)$.

(37) (3 points) If A is invertible, what is A^{-1} in terms of the adjoint (which you should define of course)?

(38) (3 points) We will now study the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$. The book tells us that there is an orthonormal basis $\{v_1, v_2\}$ for $\mathbb{R}^2$ and an orthonormal basis $\{u_1, u_2, u_3\}$ for $\mathbb{R}^3$ such that $Av_1 = \sigma_1 u_1$ and $Av_2 = \sigma_2 u_2$ (with $\sigma_1 \geq \sigma_2$). Find σ_1 and σ_2.

(39) (3 points) Find v_1 and v_2 in the previous problem.

(40) (3 points) Find u_1, u_2, and u_3 in the previous problems.